

GENERALIZED KINETIC EQUATIONS AND EFFECTIVE THERMODYNAMICS

PIERRE-HENRI CHAVANIS

*Laboratoire de Physique Théorique, Université Paul Sabatier
 118 route de Narbonne, 31062 Toulouse Cedex 4, France
 E-mail: chavanis@irsamc.ups-tlse.fr*

Abstract. We introduce a new class of nonlocal kinetic equations and nonlocal Fokker-Planck equations associated with an effective generalized thermodynamical formalism. These equations have a rich physical and mathematical structure that can describe phase transitions and blow-up phenomena. On general grounds, our formalism can have applications in different domains of physics, astrophysics, hydrodynamics and biology. We find an aesthetic connexion between topics (stars, vortices, bacteria,...) which were previously disconnected. The common point between these systems is the (attractive) long-range nature of the interactions.

1. Introduction. The statistical mechanics of systems with long-range interactions is currently a topic of active research in physics [1]. Systems with long-range interactions are numerous in nature: self-gravitating systems, two-dimensional vortices, non-neutral plasmas, metallic clusters, dipoles, fracture etc. These systems exhibit similar features such as negative specific heats, inequivalence of statistical ensembles, phase transitions, self-organization and persistence of metastable states. Among all the previous examples, self-gravitating systems and 2D vortices play a special role because they both interact via an unshielded Newtonian potential (in dimensions $D = 3$ or $D = 2$) and possess a rather similar mathematical structure [2]. Coincidentally, the chemotactic aggregation of bacteria in biology has some connexions with the collapse and organization of self-gravitating systems and 2D vortices [3, 4].

For systems with long-range interactions, the mean-field approximation turns out to be exact in a proper thermodynamic limit [5]. Therefore, their equilibrium description amounts to solving a variational problem, namely the maximization of the Boltzmann entropy $S_B[f]$ at fixed mass $M = M[f]$ and energy $E = E[f]$. In the case of quantum particles, the Boltzmann entropy is replaced by the Fermi-Dirac entropy $S_{FD}[f]$. This maximization problem determines the most probable distribution of particles at statisti-

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cal equilibrium, assuming that all the accessible microstates (with given E and M) are equiprobable. However, if, for some reason, the microstates are *not* equiprobable, other forms of entropy can emerge. This appears in the physics of complex media displaying a fractal (or multifractal) structure and exhibiting anomalous diffusion. In general, these effects are due to the existence of “hidden constraints” [6] that change the form of the transitions probabilities that we would naively expect. Motivated by this problem, we shall develop an *effective* generalized thermodynamical formalism (in μ space) associated with a larger class of entropy functionals. Specifically, we consider the maximization of a generalized entropy $S[f] = - \int C(f) d^D \mathbf{r} d^D \mathbf{v}$, where C is a convex function, at fixed mass M and energy E . Since the energy is fixed, we call this a microcanonical description. We also consider the minimization of a generalized free energy $F = E - TS$ at fixed mass M and temperature T . Since the temperature is fixed, we call this a canonical description. We discuss the inequivalence of these two descriptions when the caloric curve $\beta(E)$ presents turning points or bifurcations. This occurs in particular when the potential of interaction is long-ranged.

We also introduce a new class of relaxation equations associated with this generalized thermodynamical formalism [3, 6, 7]. We first consider a generalized class of Fokker-Planck equations extending the ordinary Kramers and Smoluchowski equations [3]. These equations have a canonical structure as they decrease a generalized free energy. When the potential of interaction is long-ranged, these equations are non-local in space and can exhibit a rich variety of behaviors including phase transitions and blow-up phenomena. In the limit of short-ranged interactions, they reduce to Cahn-Hilliard equations [7]. We also consider a generalized class of kinetic equations extending the ordinary Boltzmann and Landau equations [6]. These kinetic equations have a microcanonical structure as they conserve energy and increase a generalized entropy.

Our formalism can have applications in different domains of physics, astrophysics, fluid mechanics, biology, economy etc. with various interpretations that are not necessarily connected to thermodynamics. It is therefore important to develop a general formalism without reference to a specific context. Then, a justification has to be given in each case. For example, the maximization of the functional $S[f] = - \int C(f) d^D \mathbf{r} d^D \mathbf{v}$, where C is convex, at fixed mass and energy determines a nonlinearly dynamically stable stationary solution of the Vlasov-Poisson system for collisionless stellar systems (and 2D vortices). In this context, $S[f]$ is a H-function [8], not an entropy. Then, our generalized relaxation equations [3] can be used as numerical algorithms to construct stable stationary solutions of the Vlasov-Poisson system. On the other hand, equations similar to generalized Fokker-Planck equations appear in biology in relation with the chemotactic aggregation of bacterial populations. In any case, it is useful to develop a *thermodynamical analogy* [3] and use a vocabulary borrowed from thermodynamics. Thus, we can directly transpose the methods developed in thermodynamics to a different context.

2. Maximum entropy principle Let us consider a system of N particles in interaction and denote by $f(\mathbf{r}, \mathbf{v}, t)$ their distribution function defined such that $\int f d^D \mathbf{r} d^D \mathbf{v}$ gives

the mass of particles with position \mathbf{r} and velocity \mathbf{v} at time t . The spatial density is

$$(1) \quad \rho(\mathbf{r}, t) = \int f(\mathbf{r}, \mathbf{v}, t) d^D \mathbf{v},$$

and the total mass

$$(2) \quad M = \int \rho(\mathbf{r}, t) d^D \mathbf{r}.$$

Let $\mathbf{F}(\mathbf{r}, t) = -\nabla \Phi$ be the force (by unit of mass) experienced by a particle. We assume that the potential $\Phi(\mathbf{r}, t)$ is related to the density $\rho(\mathbf{r}, t)$ by a relation of the form

$$(3) \quad \Phi(\mathbf{r}, t) = \int \rho(\mathbf{r}', t) u(\mathbf{r} - \mathbf{r}') d^D \mathbf{r}',$$

where $u(\mathbf{r} - \mathbf{r}')$ is an arbitrary binary potential depending only on the absolute distance $|\mathbf{r} - \mathbf{r}'|$ between the particles. The energy can be expressed as

$$(4) \quad E = \int \frac{1}{2} f v^2 d^D \mathbf{r} d^D \mathbf{v} + \frac{1}{2} \int \rho \Phi d^D \mathbf{r} = K + W,$$

where K is the kinetic energy and W the potential energy. The following results remain valid if $\Phi = \Phi_{ext}(\mathbf{r})$ is a fixed external potential, in which case the potential energy reads $W = \int \rho \Phi_{ext} d^D \mathbf{r}$.

We introduce a generalized entropy of the form

$$(5) \quad S = - \int C(f) d^D \mathbf{r} d^D \mathbf{v},$$

where $C(f)$ is a convex function, i.e. $C''(f) > 0$. We are interested by the distribution function $f(\mathbf{r}, \mathbf{v})$ which maximizes the generalized entropy (5) at fixed mass and energy, i.e.

$$(6) \quad \text{Max } S[f] \quad | \quad E[f] = E, \quad M[f] = M.$$

Since the energy is fixed, we shall associate this maximization problem to a microcanonical description. Introducing Lagrange multipliers and writing the variational principle in the form

$$(7) \quad \delta S - \beta \delta E - \alpha \delta M = 0,$$

we find that the critical points of entropy at fixed mass and energy are given by

$$(8) \quad C'(f) = -\beta \epsilon - \alpha,$$

where $\epsilon = \frac{v^2}{2} + \Phi(\mathbf{r})$ is the energy of a particle by unit of mass. The Lagrange multipliers $\beta = 1/T$ and α can be interpreted as a generalized inverse temperature and a generalized chemical potential, respectively. Equation (8) can be written equivalently as

$$(9) \quad f = F(\beta \epsilon + \alpha),$$

where $F(x) = (C')^{-1}(-x)$. From the identity

$$(10) \quad f'(\epsilon) = -\beta / C''(f),$$

resulting from Eq. (8), we find that $f(\epsilon)$ is a monotonically decreasing function of energy if $\beta > 0$. The case of negative temperatures $\beta < 0$ can also be of interest depending

on the form of the function C . Explicating the relation between the potential and the density, the equilibrium distribution is determined by the integro-differential equation

$$(11) \quad C'(f) = -\beta \left\{ \frac{v^2}{2} + \int f(\mathbf{r}', \mathbf{v}') u(\mathbf{r} - \mathbf{r}') d^D \mathbf{r}' d^D \mathbf{v}' \right\} - \alpha.$$

The conservation of angular momentum $\mathbf{L} = \int f \mathbf{r} \times \mathbf{v} d^3 \mathbf{r} d^3 \mathbf{v}$ (in $D = 3$) can be easily included in the variational principle (7) by introducing an appropriate Lagrange multiplier $\beta \mathbf{\Omega}$, where $\mathbf{\Omega}$ is the angular velocity. Equation (8) remains valid provided that ϵ is replaced by the Jacobi energy $\epsilon_J = \epsilon - \mathbf{\Omega} \cdot (\mathbf{r} \times \mathbf{v}) = \frac{1}{2} \mathbf{w}^2 + \Phi_{eff}$ where $\mathbf{w} = \mathbf{v} - \mathbf{\Omega} \times \mathbf{r}$ is the relative velocity and $\Phi_{eff} = \Phi - \frac{1}{2} (\mathbf{\Omega} \times \mathbf{r})^2$ is the *effective* potential accounting for inertial forces.

We now introduce the generalized free energy

$$(12) \quad J[f] = S[f] - \beta E[f],$$

associated with the functionals (4) and (5). We are interested by the distribution function $f(\mathbf{r}, \mathbf{v})$ which maximizes the generalized free energy (12) at fixed mass and temperature, i.e.

$$(13) \quad \text{Max } J[f] = S[f] - \beta E[f] \quad | \quad M[f] = M.$$

Since the temperature is given, we shall associate this maximization problem to a canonical description. Introducing Lagrange multipliers and writing the variational principle in the form

$$(14) \quad \delta J - \alpha \delta M = 0,$$

we find that the critical points of free energy at fixed mass and temperature are given by

$$(15) \quad C'(f) = -\beta \epsilon - \alpha,$$

as in the microcanonical description.

3. Stability conditions The critical points of the variational problems (6) and (13) are the same. The equilibrium state is then obtained by solving the integro-differential equation (11) and relating the temperature β to the energy E . We can thus plot the generalized caloric curve $\beta(E)$ parameterizing the series of equilibria. We need now to determine the stability of these solutions by investigating the sign of the second order variations of S or J . In the microcanonical situation, we must select *maxima* of $S[f]$ at fixed mass and energy. The condition that f is a maximum of S at fixed mass and energy is equivalent to the condition that $\delta^2 J \equiv \delta^2 S - \beta \delta^2 E$ is negative for all perturbations that conserve mass and energy to first order. This condition can be written

$$(16) \quad \delta^2 J = - \int C''(f) \frac{(\delta f)^2}{2} d^D \mathbf{r} d^D \mathbf{v} - \frac{1}{2} \beta \int \delta \rho \delta \Phi d^D \mathbf{r} \leq 0, \\ \forall \delta f \mid \delta E = \delta M = 0.$$

The condition of stability in the canonical situation requires that f is a *maximum* of $J[f]$ at fixed mass and temperature. This is equivalent to the condition that $\delta^2 J$ is negative for all perturbations that conserve mass. This can be written

$$\delta^2 J = - \int C''(f) \frac{(\delta f)^2}{2} d^D \mathbf{r} d^D \mathbf{v} - \frac{1}{2} \beta \int \delta \rho \delta \Phi d^D \mathbf{r} \leq 0,$$

$$(17) \quad \forall \delta f \mid \delta M = 0.$$

Using Eq. (10), the functional arising in these stability criteria can be expressed as

$$(18) \quad \delta^2 J = \frac{1}{2} \beta \left\{ \int \frac{(\delta f)^2}{f'(\epsilon)} d^D \mathbf{r} d^D \mathbf{v} - \int \delta \rho \delta \Phi d^D \mathbf{r} \right\}.$$

We emphasize the importance of the first order constraints in the stability analysis. We note that canonical stability implies microcanonical stability but that the converse is wrong in general. Indeed, if inequality (17) is satisfied for all perturbations that conserve mass, it is a fortiori satisfied for perturbations that conserve mass *and* energy. Since the converse is wrong, this implies that some solutions can be stable in the microcanonical ensemble while they are unstable in the canonical one. The microcanonical and canonical descriptions are inequivalent when the caloric curve $\beta(E)$ presents turning points. This situation is well-known in the case of systems with long-range interactions such as self-gravitating systems [9]. The stability of the solutions can be decided by using the turning point criterion of Katz [10] which extends the theory of Poincaré on linear series of equilibria. It is found that a change of stability in the series of equilibria occurs in the microcanonical ensemble when the energy is extremum and in the canonical ensemble when the temperature is extremum. Stability is lost or gained depending on whether the series of equilibria $\beta(E)$ turns clockwise or anti-clockwise at that critical point. An illustration of these results is proposed in Ref. [9], in the case of self-gravitating fermions. A change of stability along a series of equilibria can also occur at a branching point [10], where the solutions bifurcate. A general classification of phase transitions for systems with long-range interactions has been proposed by Bouchet & Barré [11].

4. The free energy functional The maximization problem (13) in the canonical ensemble can be simplified. First of all, we write the free energy in the usual form

$$(19) \quad F[f] = E[f] - TS[f].$$

To solve the minimization problem

$$(20) \quad \text{Min } F[f] = E[f] - TS[f] \mid M[f] = M,$$

we shall proceed in two steps. We first minimize $F[f]$ at fixed density $\rho(\mathbf{r})$. Introducing a Lagrange multiplier $\lambda(\mathbf{r})$, we find that the global minimum $f_*(\mathbf{r}, \mathbf{v})$ of this variational problem is determined by

$$(21) \quad C'(f_*) = -\beta \left[\frac{v^2}{2} + \lambda(\mathbf{r}) \right].$$

The distribution function f_* can be written

$$(22) \quad f_* = F \left[\beta \left(\frac{v^2}{2} + \lambda(\mathbf{r}) \right) \right],$$

where $F(x) = (C')^{-1}(-x)$. We define the density and the pressure by

$$(23) \quad \rho = \int f d^D \mathbf{v}, \quad p = \frac{1}{D} \int f v^2 d^D \mathbf{v}.$$

Substituting Eq. (22) in the foregoing expressions, we find that

$$(24) \quad \rho = \frac{1}{\beta^{D/2}} g(\beta\lambda), \quad p = \frac{1}{\beta^{\frac{D+2}{2}}} h(\beta\lambda),$$

with

$$(25) \quad g(x) = 2^{\frac{D-2}{2}} S_D \int_0^{+\infty} F(x+t) t^{\frac{D-2}{2}} dt,$$

$$(26) \quad h(x) = \frac{1}{D} 2^{\frac{D}{2}} S_D \int_0^{+\infty} F(x+t) t^{\frac{D}{2}} dt,$$

where S_D is the surface of a unit sphere in D dimensions. Eliminating λ between the foregoing expressions, we find that the fluid is *barotropic*, in the sense that $p = p(\rho)$ where the equation of state is entirely specified by $C(f)$. We can now express the free energy (19) as a functional of ρ by using $F[\rho] = F[f_*]$. The energy (4) is simply given by

$$(27) \quad E = \frac{D}{2} \int p d^D \mathbf{r} + \frac{1}{2} \int \rho \Phi d^D \mathbf{r}.$$

On the other hand, the entropy (5) can be written

$$(28) \quad S = -\frac{2^{\frac{D-2}{2}} S_D}{\beta^{D/2}} \int d^D \mathbf{r} \int_0^{+\infty} C[F(t + \beta\lambda)] t^{\frac{D-2}{2}} dt.$$

Integrating by parts and using $C'[F(x)] = -x$, we find that

$$(29) \quad S = -\frac{2^{D/2} S_D}{D \beta^{D/2}} \int d^D \mathbf{r} \int_0^{+\infty} F'(t + \beta\lambda) (t + \beta\lambda) t^{D/2} dt.$$

Integrating by parts one more time and using Eqs. (24), (25) and (26), we finally obtain

$$(30) \quad S = \frac{D+2}{2} \beta \int p d^D \mathbf{r} + \beta \int \lambda \rho d^D \mathbf{r}.$$

Collecting all the previous results, the free energy (19) becomes

$$(31) \quad F[\rho] = - \int \rho \left(\lambda + \frac{p}{\rho} \right) d^D \mathbf{r} + \frac{1}{2} \int \rho \Phi d^D \mathbf{r}.$$

Finally, using the relation $h'(x) = -g(x)$ obtained from Eqs. (25) and (26) by a simple integration by parts, it is easy to check that Eq. (24) implies

$$(32) \quad \lambda + \frac{p}{\rho} = - \int_0^\rho \frac{p(\rho')}{\rho'^2} d\rho'.$$

Hence, the free energy can be written more explicitly as

$$(33) \quad F[\rho] = \int \rho \int_0^\rho \frac{p(\rho')}{\rho'^2} d\rho' d^D \mathbf{r} + \frac{1}{2} \int \rho \Phi d^D \mathbf{r}.$$

We are led therefore to study the minimization problem

$$(34) \quad \text{Min } F[\rho] \quad | \quad M[\rho] = M,$$

for the free energy functional (33). Writing the first order variations in the form

$$(35) \quad \delta F - \alpha \delta M = 0,$$

we find that the critical points of free energy satisfy the condition of hydrostatic balance

$$(36) \quad \nabla p = -\rho \nabla \Phi,$$

and that $\lambda = \Phi + \alpha/\beta$. We can easily check that the condition of hydrostatic equilibrium is directly implied by the relation $f = f(\epsilon)$ derived in Sec. 2. This immediately results from the identities

$$(37) \quad \rho = \frac{1}{D} \int f \frac{\partial \mathbf{v}}{\partial \mathbf{v}} d^D \mathbf{v} = -\frac{1}{D} \int \frac{\partial f}{\partial \mathbf{v}} \cdot \mathbf{v} d^D \mathbf{v} = -\frac{1}{D} \int f'(\epsilon) v^2 d^D \mathbf{v},$$

$$(38) \quad \nabla p = \frac{1}{D} \int \frac{\partial f}{\partial \mathbf{r}} v^2 d^D \mathbf{v} = \frac{1}{D} \int f'(\epsilon) \nabla \Phi v^2 d^D \mathbf{v} = -\rho \nabla \Phi.$$

Explicating the relation between the potential and the density, the condition of hydrostatic equilibrium can be written in the form of an integro-differential equation

$$(39) \quad \frac{p'(\rho)}{\rho} \nabla \rho = -\nabla \int \rho(\mathbf{r}') u(\mathbf{r} - \mathbf{r}') d^D \mathbf{r}'.$$

This equation determines the equilibrium solutions (critical points of F). Their stability (minima of F) is determined by the condition

$$(40) \quad \delta^2 F = \int \frac{p'(\rho)}{2\rho} (\delta\rho)^2 d^D \mathbf{r} + \frac{1}{2} \int \delta\rho \delta\Phi d^D \mathbf{r} \geq 0, \\ \forall \delta\rho \mid \delta M = 0.$$

Using the condition of hydrostatic equilibrium, the above functional can be rewritten as

$$(41) \quad \delta^2 F = \frac{1}{2} \int \frac{(\delta\rho)^2}{-\rho'(\Phi)} d^D \mathbf{r} + \frac{1}{2} \int \delta\rho \delta\Phi d^D \mathbf{r}.$$

5. Examples of entropy functionals Among all functionals of the form (5), some have been discussed in detail in the literature. The most famous functional is the Boltzmann entropy

$$(42) \quad S_B[f] = - \int f \ln f d^D \mathbf{r} d^D \mathbf{v}.$$

It leads to the isothermal distribution

$$(43) \quad f = A e^{-\beta\epsilon}.$$

The corresponding distribution in physical space is the Boltzmann distribution

$$(44) \quad \rho = A' e^{-\beta\Phi}, \quad A' = \left(\frac{2\pi}{\beta} \right)^{D/2} A.$$

The distribution function (43) leads to the classical equation of state

$$(45) \quad p = \rho T,$$

and to the free energy

$$(46) \quad F[\rho] = T \int \rho \ln \rho d^D \mathbf{r} + \frac{1}{2} \int \rho \Phi d^D \mathbf{r}.$$

Closely related to the Boltzmann entropy is the Fermi-Dirac entropy

$$(47) \quad S_{FD}[f] = - \int \left\{ \frac{f}{\eta_0} \ln \frac{f}{\eta_0} + \left(1 - \frac{f}{\eta_0} \right) \ln \left(1 - \frac{f}{\eta_0} \right) \right\} d^D \mathbf{r} d^D \mathbf{v},$$

which leads to the Fermi-Dirac distribution function

$$(48) \quad f = \frac{\eta_0}{1 + \lambda e^{\beta\eta_0\epsilon}}.$$

The Fermi-Dirac distribution function (48) satisfies the constraint $f \leq \eta_0$ which is related to Pauli's exclusion principle in quantum mechanics. The isothermal distribution function (43) is recovered in the non-degenerate limit $f \ll \eta_0$. The distribution in physical space associated with the Fermi-Dirac statistics is

$$(49) \quad \rho = \frac{\eta_0 S_D 2^{\frac{D}{2}-1}}{\beta^{D/2}} I_{\frac{D}{2}-1}(\lambda e^{\beta\Phi}),$$

where I_n is the Fermi integral

$$(50) \quad I_n(t) = \int_0^{+\infty} \frac{x^n}{1 + te^x} dx.$$

The distribution function (48) leads to the quantum equation of state given in parametric form as

$$(51) \quad \rho = \frac{\eta_0 S_D 2^{\frac{D}{2}-1}}{\beta^{D/2}} I_{\frac{D}{2}-1}(\lambda'), \quad p = \frac{\eta_0 S_D 2^{\frac{D}{2}}}{D \beta^{\frac{D}{2}+1}} I_{\frac{D}{2}}(\lambda').$$

Recently, there was a considerable interest in physics for functionals of the form

$$(52) \quad S_q[f] = -\frac{1}{q-1} \int (f^q - f) d^D \mathbf{r} d^D \mathbf{v},$$

where q is a real number. Such functionals introduced by Tsallis [12] are called q -entropies. They lead to polytropic distribution functions

$$(53) \quad f = \left[\mu - \frac{(q-1)\beta}{q} \epsilon \right]^{\frac{1}{q-1}}.$$

The index n of the polytrope is related to the parameter q by the relation $n = D/2 + 1/(q-1)$ [13]. Isothermal distribution functions are recovered in the limit $q \rightarrow 1$ (i.e. $n \rightarrow +\infty$). The distribution function (53) leads to the polytropic equation of state

$$(54) \quad p = K \rho^\gamma, \quad \gamma = 1 + \frac{1}{n},$$

with

$$(55) \quad K = \frac{1}{n+1} \left\{ 2^{\frac{D}{2}-1} S_D A B\left(\frac{D}{2}, n+1 - \frac{D}{2}\right) \right\}^{-1/n},$$

where $A = [(q-1)\beta/q]^{\frac{1}{q-1}}$ and $B(n, m)$ is the Beta function. The corresponding free energy can be written

$$(56) \quad F[\rho] = \frac{K}{\gamma-1} \int (\rho^\gamma - \rho) d^D \mathbf{r} + \frac{1}{2} \int \rho \Phi d^D \mathbf{r},$$

and the density in physical space is

$$(57) \quad \rho = \left[\lambda - \frac{\gamma-1}{K\gamma} \Phi \right]^{\frac{1}{\gamma-1}}.$$

We note that a polytropic distribution with index q in phase space yields a polytropic distribution with index $\gamma = 1 + 2(q-1)/[2 + D(q-1)]$ in physical space. In this sense, Tsallis distributions are stable laws. By comparing Eqs. (53) and (57) or Eqs. (19), (52) and (56) we note that K plays the same role in physical space as the temperature $T = 1/\beta$ in phase space. It is sometimes called a ‘‘polytropic temperature’’.

We have started a systematic study of the variational problems (6), (13) and (34) by considering first the gravitational interaction in D dimensions. The case of isothermal self-gravitating systems associated with the Boltzmann entropy has been considered in [14]. The case of self-gravitating fermions associated with the Fermi-Dirac entropy has been considered in [9, 15]. The case of polytropic self-gravitating systems associated with the Tsallis entropy has been considered in [13].

6. Physical applications The maximization problems discussed previously can have various physical applications (not necessarily related to thermodynamics) that we briefly mention.

(i) *Statistical mechanics:* The variational principles (6), (13) and (34) determine the statistical equilibrium states of systems with long-range interactions, such as self-gravitating systems. In that case, $S[f]$ is the Boltzmann entropy (42) for classical particles (e.g., stars in globular clusters) [16] or the Fermi-Dirac entropy (47) for fermions (e.g., massive neutrinos in dark matter models) [9]. The microcanonical situation (6) applies to isolated Hamiltonian systems such as stellar systems [17]. The canonical situation (13) or (34) applies to systems in contact with a heat bath imposing its temperature, like for the interstellar medium [18]. This is also the proper statistical description of a gas of self-gravitating Brownian particles [19, 20]. This discussion remains valid for other types of long-range interactions.

(ii) *Vlasov equation:* The variational principle (6) determines nonlinearly dynamically stable stationary solutions of the Vlasov-Poisson system

$$(58) \quad \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0,$$

$$(59) \quad \Delta \Phi = 4\pi G \int f d^3 \mathbf{v},$$

which describes collisionless stellar systems. These robust organized states can emerge as a result of a violent relaxation [22, 23]. In that context, $S[f]$ is called a H-function [8]. Boltzmann and Tsallis functionals are particular H-functions (not true entropies in that context) associated with isothermal and polytropic stellar systems [21, 3]. Note that the criterion of nonlinear dynamical stability (6)-(16) is more refined than the stability criterion (13)-(17). These results on nonlinear dynamical stability remain valid when the force \mathbf{F} is related to the distribution function f by a relation of the general form (3).

(iii) *Euler-Jeans equation:* The variational principle (34) determines the nonlinear dynamical stability of stationary solutions of the Euler-Jeans-Poisson system

$$(60) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

$$(61) \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p - \nabla \Phi, \quad \Delta \Phi = 4\pi G \rho,$$

describing barotropic stars. In that context, $\mathcal{W}[\rho] = F[\rho] + \int \rho \frac{u^2}{2} d^3 \mathbf{r}$ represents the energy of the star. The first term in Eq. (33) is the internal energy of a barotropic gas and the second term is the gravitational energy (a kinetic term has also to be introduced in the energy functional \mathcal{W}). Since the stability condition (34) implies (6), we note that a

stellar system is stable with respect to the Vlasov equation whenever the corresponding barotropic star with the same equation of state is stable with respect to the Euler-Jeans equations (but the reciprocal is wrong in general) [21]. In astrophysics, this is known as the Antonov's first law [24].

(iv) *Generalized thermodynamics*: The variational principles (6), (13) and (34) determine the generalized thermodynamical stability of complex systems exhibiting anomalous diffusion. Such systems have a complicated phase space structure (fractal, multifractal,...) due to the action of microscopic constraints (hidden constraints) that are often difficult to formalize. These systems can be described by *effective* equations resembling generalized Boltzmann and Fokker-Planck equations [3, 6]. The variational principles (6), (13) and (34) also determine the linear dynamical stability of stationary solutions of these equations. Accordingly, there is a close connexion between thermodynamical and dynamical stability [3].

7. Generalized Kramers equation We shall introduce formally a relaxation equation associated with the minimization problem (20). We write this equation in the form

$$(62) \quad \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \cdot \left[D \beta f \frac{\partial}{\partial \mathbf{v}} \left(\frac{\delta F}{\delta f} \right) \right],$$

where $\delta/\delta f$ denotes a functional derivative. By construction, this equation conserves mass since the right hand side is the divergence of a current $-\mathbf{J}_f$. Using the expression of the free energy (19) with Eqs. (4) and (5), we obtain the generalized Fokker-Planck equation

$$(63) \quad \frac{\partial f}{\partial t} + Lf = \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ D \left[f C''(f) \frac{\partial f}{\partial \mathbf{v}} + \beta f \mathbf{v} \right] \right\},$$

where L is the advection operator. Morphologically, Eq. (63) can be viewed as a generalized non-local Kramers equation [25]. It describes the dynamics (in phase space) of a system of Langevin particles in interaction governed by a generalized class of stochastic processes [3, 5]:

$$(64) \quad \frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i, \quad \frac{d\mathbf{v}_i}{dt} = -\xi \mathbf{v}_i - \nabla_i U(\mathbf{r}_1, \dots, \mathbf{r}_N) + \sqrt{2Df_i \left[\frac{C(f_i)}{f_i} \right]'} \mathbf{R}_i(t),$$

where $-\xi \mathbf{v}_i$ is a friction force and $U(\mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{i < j} u(\mathbf{r}_i - \mathbf{r}_j)$ is a potential of interaction. The last term is a generalized stochastic force. The usual white noise term $\mathbf{R}_i(t)$, satisfying $\langle \mathbf{R}_i(t) \rangle = \mathbf{0}$ and $\langle R_{a,i}(t) R_{b,j}(t') \rangle = \delta_{ij} \delta_{ab} \delta(t - t')$, where $a, b = 1, \dots, D$ refer to the coordinates of space and $i, j = 1, \dots, N$ to the particles, is multiplied by a function that can depend on the local (macroscopic) distribution function $f_i \equiv f(\mathbf{r}_i, \mathbf{v}_i, t)$. When $C(f) = f \ln f$, which is related to the Boltzmann entropy (42), Eq. (64) reduces to the usual Langevin equations

$$(65) \quad \frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i, \quad \frac{d\mathbf{v}_i}{dt} = -\xi \mathbf{v}_i - \nabla_i U(\mathbf{r}_1, \dots, \mathbf{r}_N) + \sqrt{2D} \mathbf{R}_i(t).$$

This stochastic process describes a system of Brownian particles in interaction. More generally, the term in front of $\mathbf{R}_i(t)$ in Eq. (64) can be interpreted as a multiplicative noise since it depends on \mathbf{r}, \mathbf{v} . Note that it depends on \mathbf{r}, \mathbf{v} through the distribution function $f(\mathbf{r}, \mathbf{v}, t)$. Therefore, there is a back-reaction from the macrodynamics, leading

to a situation of anomalous diffusion. In that context, the generalized Kramers equation (63) can be derived from the N-body Fokker-Planck equation by using a Kramers-Moyal expansion and a mean-field approximation [4, 5]. The first term in Eq. (63) is a generalized diffusion (depending on the distribution function) and the second term is a friction. The friction coefficient $\xi = D\beta$ satisfies a generalized Einstein relation. Note that D can depend on \mathbf{r} , \mathbf{v} and t without altering the general properties of the equation. We can use this indetermination to write the generalized Kramers equation in the alternative form

$$(66) \quad \frac{\partial f}{\partial t} + Lf = \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ D' \left[\frac{\partial f}{\partial \mathbf{v}} + \frac{\beta}{C''(f)} \mathbf{v} \right] \right\},$$

which will have the same general properties as Eq. (63). This equation involves an ordinary diffusion and a nonlinear friction. Equation (66) can be deduced from Eq. (63) by the substitution $D' = DfC''(f)$. One of these two forms will be preferred depending on the situation contemplated.

The generalized Kramers equation (63) can also be obtained from a variational principle by maximizing the rate of free energy dissipation \dot{F} at fixed mass and temperature [3]. Therefore, Eq. (63) satisfies a canonical H-theorem $\dot{F} \leq 0$, provided that $D \geq 0$. Indeed, an explicit calculation yields

$$(67) \quad \dot{F} = - \int \frac{DT}{f} \left[fC''(f) \frac{\partial f}{\partial \mathbf{v}} + \beta f \mathbf{v} \right]^2 d^D \mathbf{r} d^D \mathbf{v} \leq 0.$$

This shows that F is the Lyapunov functional of the generalized Kramers equation. Now, at equilibrium $\dot{F} = 0$, so that according to Eq. (67),

$$(68) \quad \frac{\partial C'(f)}{\partial \mathbf{v}} + \beta \mathbf{v} = \mathbf{0}.$$

Integrating with respect to \mathbf{v} , we get

$$(69) \quad C'(f) = -\beta \frac{v^2}{2} + A(\mathbf{r}).$$

The cancellation of the advective term $Lf = 0$ in Eq. (63) combined with Eq. (69) implies that $f = f(\epsilon)$ and $\nabla A = -\beta \nabla \Phi$. Therefore, $A(\mathbf{r}) = -\beta \Phi(\mathbf{r}) - \alpha$ and we recover Eq. (15). This shows that a stationary solution of Eq. (63) extremizes the free energy F at fixed mass and temperature. In addition, only *minima* of F at fixed M and T are linearly stable with respect to the generalized Kramers equation (63). Indeed, considering the linear stability of a stationary solution of Eq. (63), we can derive the general relation

$$(70) \quad 2\lambda \delta^2 F = \delta^2 \dot{F} \leq 0,$$

connecting the growth rate λ of the perturbation $\delta f \sim e^{\lambda t}$ to the second order variations of the free energy F and the second order variations of the rate of free energy production $\delta^2 \dot{F} \leq 0$ [3]. Since the product $\lambda \delta^2 F$ is negative, we conclude that a stationary solution of the generalized Kramers equation (63) is linearly stable ($\lambda < 0$) if, and only if, it is a *minimum* of free energy F at fixed mass and temperature. This aesthetic formula shows the equivalence between dynamical and thermodynamical stability for the generalized Kramers equation. Therefore, it only selects *minima* of F , not maxima or saddle points.

8. Generalized Smoluchowski equation We shall now introduce formally a relaxation equation associated with the minimization problem (34). We write this equation in the form

$$(71) \quad \frac{\partial \rho}{\partial t} = \nabla \cdot \left[\frac{1}{\xi} \rho \nabla \left(\frac{\delta F}{\delta \rho} \right) \right].$$

Using the expression of the free energy (33), we obtain the generalized Fokker-Planck equation

$$(72) \quad \frac{\partial \rho}{\partial t} = \nabla \cdot \left[\frac{1}{\xi} (\nabla p + \rho \nabla \Phi) \right].$$

Morphologically, Eq. (72) can be viewed as a generalized non-local Smoluchowski equation [25]. It can be obtained from a variational principle by maximizing the rate of free energy dissipation \dot{F} at fixed mass and temperature [3]. Therefore, Eq. (72) satisfies a canonical H-theorem $\dot{F} \leq 0$. Indeed, an explicit calculation yields

$$(73) \quad \dot{F} = - \int \frac{1}{\xi \rho} (\nabla p + \rho \nabla \Phi)^2 d^D \mathbf{r} \leq 0.$$

This shows that F is the Lyapunov functional of the generalized Smoluchowski equation. Now, at equilibrium $\dot{F} = 0$, and we recover the condition of hydrostatic balance (36). It is also possible to show [3] that only *minima* of free energy F are linearly stable with respect to the generalized Smoluchowski equation (72).

Explicating the relation between the potential and the density, the generalized Smoluchowski equation (72) can be written

$$(74) \quad \frac{\partial \rho}{\partial t} = \nabla \cdot \left\{ \frac{1}{\xi} \left[p'(\rho) \nabla \rho + \rho \nabla \int u(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}', t) d^D \mathbf{r}' \right] \right\}.$$

The Lyapunov functional associated with this equation is the free energy

$$(75) \quad F[\rho] = \int \rho \int_0^\rho \frac{p(\rho')}{\rho'^2} d\rho' d^D \mathbf{r} + \frac{1}{2} \int \rho(\mathbf{r}, t) u(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}', t) d^D \mathbf{r} d^D \mathbf{r}'.$$

The stationary states are determined by the integro-differential equation

$$(76) \quad \frac{p'(\rho)}{\rho} \nabla \rho = - \nabla \int \rho(\mathbf{r}') u(\mathbf{r} - \mathbf{r}') d^D \mathbf{r}'.$$

Equation (74) generalizes the usual Smoluchowski equation in two respects. First, the coefficient of diffusion $\frac{1}{\xi} p'(\rho)$ is an arbitrary function of the density ρ associated with a generalized entropy functional (first term in Eq. (75)). Second, this equation is non-local due to the presence of an arbitrary binary potential of interaction $u(\mathbf{r} - \mathbf{r}')$ in the energy functional (second term in Eq. (75)). This equation possesses therefore a very rich mathematical and physical structure.

We have started a systematic study of the generalized Smoluchowski equation (74) by considering first the gravitational interaction in D dimensions. The case of an isothermal equation of state corresponding to the Boltzmann free energy has been considered in [19, 14, 26, 27]. It describes a system of self-gravitating Brownian particles (see a review in [20]). The case of a quantum equation of state associated with the Fermi-Dirac free energy has been considered in [28, 4]. It describes a system of self-gravitating Brownian fermions. Finally the case of a polytropic equation of state associated with Tsallis free

energy has been considered in [13]. It describes a system of self-gravitating Langevin particles experiencing anomalous diffusion.

9. The method of moments The Smoluchowski equation can be derived from the Kramers equation in a high friction limit $\xi \rightarrow +\infty$, or equivalently for large times $t \gg \xi^{-1}$. This can be shown easily by using a method of moments. Integrating Eq. (63) over velocity, we get the continuity equation

$$(77) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

where $\mathbf{u} = (1/\rho) \int f \mathbf{v} d^3 \mathbf{v}$ is the local velocity. Multiplying Eq. (63) by \mathbf{v} and integrating over velocity, we get the momentum equation

$$(78) \quad \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial}{\partial x_j} P_{ij} + \rho \frac{\partial \Phi}{\partial x_i} = - \int D \left[f C''(f) \frac{\partial f}{\partial v_i} + \beta f v_i \right] d^3 \mathbf{v},$$

where $P_{ij} = \int f w_i w_j d^3 \mathbf{v}$ is the pressure tensor and $\mathbf{w} = \mathbf{v} - \mathbf{u}$ the relative velocity. Introducing the notation $\phi(f) = \int^f x C''(x) dx$, the first term in the right hand side can be rewritten $\partial \phi(f) / \partial \mathbf{v}$ and, since it is a gradient of a function, it vanishes by integration. We are left therefore with

$$(79) \quad \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial}{\partial x_j} P_{ij} + \rho \frac{\partial \Phi}{\partial x_i} = -\xi \rho u_i.$$

In the high friction limit $\xi = D\beta \rightarrow +\infty$, assuming β of order unity, the term in bracket in Eq. (63) must vanish so that the distribution function satisfies

$$(80) \quad C'(f) = -\beta \left[\frac{v^2}{2} + \lambda(\mathbf{r}, t) \right] + O(\xi^{-1}),$$

where $\lambda(\mathbf{r}, t)$ is related to the spatial density $\rho(\mathbf{r}, t)$ through the relation

$$(81) \quad \rho = \int f d^D \mathbf{v}.$$

We find therefore that $\mathbf{u} = O(\xi^{-1})$ and $P_{ij} = p \delta_{ij} + O(\xi^{-1})$ where $p(\mathbf{r}, t)$ is the local pressure

$$(82) \quad p = \frac{1}{D} \int f v^2 d^D \mathbf{v},$$

determined by Eq. (80). As in Sec. 4, the fluid is *barotropic*, i.e. $p(\mathbf{r}, t) = p[\rho(\mathbf{r}, t)]$. To first order in ξ^{-1} , the momentum equation (79) implies that

$$(83) \quad \rho \mathbf{u} = -\frac{1}{\xi} (\nabla p + \rho \nabla \Phi) + O(\xi^{-2}).$$

Inserting the relation (83) in the continuity equation (77), we get the generalized Smoluchowski equation (72). The generalized Smoluchowski equation, as well as the first order correction to the distribution function $f(\mathbf{r}, \mathbf{v}, t)$, can also be obtained from a formal Chapman-Enskog expansion [7].

10. Generalized kinetic equations The Kramers equation and the Smoluchowski equation have a canonical structure in which the temperature is fixed. We shall now

introduce a generalized kinetic equation possessing a microcanonical structure in which the energy is fixed [3, 6]. This is the generalized Landau equation

$$(84) \quad \frac{\partial f}{\partial t} + Lf = \frac{\partial}{\partial v^\mu} \int d^3 \mathbf{v}_1 K^{\mu\nu} f f_1 \left\{ C''(f) \frac{\partial f}{\partial v^\nu} - C''(f_1) \frac{\partial f_1}{\partial v_1^\nu} \right\},$$

with

$$(85) \quad K^{\mu\nu} = \frac{A}{u} \left(\delta^{\mu\nu} - \frac{u^\mu u^\nu}{u^2} \right),$$

where $\mathbf{u} = \mathbf{v}_1 - \mathbf{v}$ is the relative velocity and A is a constant. We can also consider the alternative form

$$(86) \quad \frac{\partial f}{\partial t} + Lf = \frac{\partial}{\partial v^\mu} \int d^3 \mathbf{v}_1 K^{\mu\nu} \left\{ \frac{1}{C''(f_1)} \frac{\partial f}{\partial v^\nu} - \frac{1}{C''(f)} \frac{\partial f_1}{\partial v_1^\nu} \right\}.$$

The generalized Landau equation can be derived from a generalized Boltzmann equation in a weak deflexion approximation. In turn, the generalized Kramers equation (63) can be derived from the generalized Landau equation in a thermal bath approximation [6]. The generalized Landau equation satisfies a H-theorem ($\dot{S} \geq 0$) for the generalized entropy (5). The entropy (5) plays therefore the role of a Lyapunov functional. At equilibrium, $\dot{S} = 0$, and we obtain the distribution (8). In addition, it can be shown [6] that only *maxima* of S at fixed M and E are linearly stable with respect to the generalized Landau equation.

Generalized kinetic equations such as (84) appear when the transition probabilities are different from the form that we would naively expect due to the action of microscopic constraints (hidden constraints). One particular case is when the particles are fermions. In that case, the “hidden constraints” correspond to the Pauli exclusion principle that prevents two particles with equal spin to occupy the same phase space cell. This exclusion has been explained by quantum mechanics. More generally, the “hidden constraints” may not have necessarily a fundamental origin.

11. Particular examples It can be of interest to discuss some special cases explicitly. For the Boltzmann entropy (42), $C''(f) = 1/f$ and Eq. (63) reduces to the ordinary Kramers equation

$$(87) \quad \frac{\partial f}{\partial t} + Lf = \frac{\partial}{\partial \mathbf{v}} \cdot \left[D \left(\frac{\partial f}{\partial \mathbf{v}} + \beta f \mathbf{v} \right) \right].$$

The corresponding equation in physical space, obtained in the high friction limit, is the ordinary Smoluchowski equation

$$(88) \quad \frac{\partial \rho}{\partial t} = \nabla \cdot \left[\frac{1}{\xi} (T \nabla \rho + \rho \nabla \Phi) \right].$$

Finally, Eq. (84) leads to the ordinary Landau equation

$$(89) \quad \frac{\partial f}{\partial t} + Lf = \frac{\partial}{\partial v^\mu} \int d^3 \mathbf{v}_1 K^{\mu\nu} \left(f_1 \frac{\partial f}{\partial v^\nu} - f \frac{\partial f_1}{\partial v_1^\nu} \right).$$

For the Fermi-Dirac entropy (47), $C''(f) = 1/f(\eta_0 - f)$. In order to avoid the divergence of the term $f C''(f)$ as $f \rightarrow \eta_0$, it is appropriate to consider the alternative form

(66) of the generalized Kramers equation. This yields

$$(90) \quad \frac{\partial f}{\partial t} + Lf = \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ D' \left[\frac{\partial f}{\partial \mathbf{v}} + \beta f (\eta_0 - f) \mathbf{v} \right] \right\}.$$

The corresponding equation in physical space is given by Eq. (72) with the equation of state (51). In fact, when the Kramers equation is written in the form (66), the coefficient in factor of the diffusion current in Eq. (72) is *not* $1/\xi$ but is more complex. We refer to [7] for a detailed discussion of this subtle point. Finally, Eq. (86) leads to the fermionic Landau equation

$$(91) \quad \frac{\partial f}{\partial t} + Lf = \frac{\partial}{\partial v^\mu} \int d^3 \mathbf{v}_1 K^{\mu\nu} \left\{ f_1 (\eta_0 - f_1) \frac{\partial f}{\partial v^\nu} - f (\eta_0 - f) \frac{\partial f_1}{\partial v_1^\nu} \right\}.$$

For the Tsallis entropy (52), $C''(f) = qf^{q-2}$ and Eq. (63) has the form of a nonlinear Kramers equation

$$(92) \quad \frac{\partial f}{\partial t} + Lf = \frac{\partial}{\partial \mathbf{v}} \left[D \left(\frac{\partial f^q}{\partial \mathbf{v}} + \beta f \mathbf{v} \right) \right].$$

The corresponding equation in physical space is the nonlinear Smoluchowski equation

$$(93) \quad \frac{\partial \rho}{\partial t} = \nabla \left[\frac{1}{\xi} (K \nabla \rho^q + \rho \nabla \Phi) \right].$$

Finally, Eq. (84) leads to the q -Landau equation

$$(94) \quad \frac{\partial f}{\partial t} + Lf = \frac{\partial}{\partial v^\mu} \int d^3 \mathbf{v}_1 K^{\mu\nu} \left(f_1 \frac{\partial f^q}{\partial v^\nu} - f \frac{\partial f_1^q}{\partial v_1^\nu} \right).$$

12. Maximum entropy principle in physical space We now consider a system of N particles in interaction with total mass

$$(95) \quad M = \int \rho(\mathbf{r}, t) d^D \mathbf{r},$$

and energy

$$(96) \quad E = \frac{1}{2} \int \rho \Phi d^D \mathbf{r},$$

where the potential $\Phi(\mathbf{r}, t)$ is related to the density $\rho(\mathbf{r}, t)$ by a relation of the form

$$(97) \quad \Phi(\mathbf{r}, t) = \int \rho(\mathbf{r}', t) u(\mathbf{r} - \mathbf{r}') d^D \mathbf{r}'.$$

We introduce a generalized entropy in physical space

$$(98) \quad S = - \int C(\rho) d^D \mathbf{r},$$

where $C(\rho)$ is a convex function, i.e. $C''(\rho) > 0$. We are interested by the distribution $\rho(\mathbf{r})$ which maximizes the generalized entropy (98) at fixed mass and energy, i.e.

$$(99) \quad \text{Max } S[\rho] \quad | \quad E[\rho] = E, \quad M[\rho] = M.$$

Since the energy is fixed, we shall associate this maximization problem to a microcanonical description. Introducing Lagrange multipliers and writing the variational principle in the form

$$(100) \quad \delta S - \beta \delta E - \alpha \delta M = 0,$$

we find that the critical points of entropy at fixed mass and energy are given by

$$(101) \quad C'(\rho) = -\beta\Phi - \alpha.$$

The Lagrange multipliers $\beta = 1/T$ and α can be interpreted as a generalized inverse temperature and a generalized chemical potential, respectively. Equation (101) can be written equivalently as

$$(102) \quad \rho = F(\beta\Phi + \alpha),$$

where $F(x) = (C')^{-1}(-x)$. From the identity

$$(103) \quad \rho'(\Phi) = -\beta/C''(\rho),$$

resulting from Eq. (101), $\rho(\Phi)$ is a monotonically decreasing function of Φ if $\beta > 0$ and a monotonically increasing function of Φ if $\beta < 0$. Explicating the relation between the potential and the density, we find that the equilibrium distribution is determined by the integro-differential equation

$$(104) \quad C'(\rho) = -\beta \int \rho(\mathbf{r}') u(\mathbf{r} - \mathbf{r}') d^D \mathbf{r}' - \alpha.$$

We now introduce the generalized free energy

$$(105) \quad J[\rho] = S[\rho] - \beta E[\rho],$$

associated with the functionals (96) and (98). We are interested by the density $\rho(\mathbf{r})$ which maximizes the generalized free energy (105) at fixed mass and temperature, i.e.

$$(106) \quad \text{Max } J[\rho] = S[\rho] - \beta E[\rho] \quad | \quad M[\rho] = M.$$

Since the temperature is given, we shall associate this maximization problem to a canonical description. Introducing Lagrange multipliers and writing the variational principle in the form

$$(107) \quad \delta J - \alpha \delta M = 0,$$

we find that the critical points of free energy at fixed mass and temperature are given by

$$(108) \quad C'(\rho) = -\beta\Phi - \alpha,$$

as in the microcanonical description.

13. Stability conditions In the microcanonical situation, we must select *maxima* of $S[\rho]$ at fixed mass and energy. The condition that ρ is a maximum of S at fixed mass and energy is equivalent to the condition that $\delta^2 J \equiv \delta^2 S - \beta \delta^2 E$ is negative for all perturbations that conserve mass and energy to first order. This condition can be written

$$(109) \quad \delta^2 J = - \int C''(\rho) \frac{(\delta\rho)^2}{2} d^D \mathbf{r} - \frac{1}{2} \beta \int \delta\rho \delta\Phi d^D \mathbf{r} \leq 0, \\ \forall \delta\rho \mid \delta E = \delta M = 0.$$

The condition of stability in the canonical situation requires that ρ is a *maximum* of $J[\rho]$ at fixed mass and temperature. This is equivalent to the condition that $\delta^2 J$ is negative for all perturbations that conserve mass. This can be written

$$\delta^2 J = - \int C''(\rho) \frac{(\delta\rho)^2}{2} d^D \mathbf{r} - \frac{1}{2} \beta \int \delta\rho \delta\Phi d^D \mathbf{r} \leq 0,$$

$$(110) \quad \forall \delta\rho \mid \delta M = 0.$$

Using Eq. (103), the functional arising in these stability criteria can be expressed as

$$(111) \quad \delta^2 J = \frac{1}{2} \beta \left\{ \int \frac{(\delta\rho)^2}{\rho'(\Phi)} d^D \mathbf{r} - \int \delta\rho \delta\Phi d^D \mathbf{r} \right\}.$$

The discussion on the inequivalence of the microcanonical and canonical descriptions is the same as the one given in Sec. 3.

14. Examples of entropy functionals

The Boltzmann entropy in physical space

$$(112) \quad S_B[\rho] = - \int \rho \ln \rho d^D \mathbf{r},$$

leads to the isothermal distribution

$$(113) \quad \rho = A e^{-\beta\Phi}.$$

The Fermi-Dirac entropy in physical space

$$(114) \quad S_{FD}[\rho] = - \int \left\{ \frac{\rho}{\sigma_0} \ln \frac{\rho}{\sigma_0} + \left(1 - \frac{\rho}{\sigma_0} \right) \ln \left(1 - \frac{\rho}{\sigma_0} \right) \right\} d^D \mathbf{r},$$

leads to the Fermi-Dirac distribution

$$(115) \quad \rho = \frac{\sigma_0}{1 + \lambda e^{\beta\sigma_0\Phi}}.$$

This distribution satisfies the constraint $\rho \leq \sigma_0$ which puts an upper bound on the local density of particles. It usually arises when one considers finite size effects or when one introduces a lattice model in physical space preventing two particles to occupy the same site. The isothermal distribution (113) is recovered in the dilute limit $\rho \ll \sigma_0$.

The Tsallis entropy

$$(116) \quad S_q[\rho] = - \frac{1}{q-1} \int (\rho^q - \rho) d^D \mathbf{r},$$

where q is a real number, leads to the polytropic distribution

$$(117) \quad \rho = \left[\mu - \frac{(q-1)\beta}{q} \Phi \right]^n, \quad n = \frac{1}{q-1}.$$

The isothermal distribution (113) is recovered for $q \rightarrow 1$, i.e. $n \rightarrow +\infty$.

15. Physical applications The maximization problems discussed previously can have various physical applications (not necessarily related to thermodynamics) that we briefly mention.

(i) *Statistical mechanics of point vortices:* The variational problems (99) and (106) arise in the statistical mechanics of 2D point vortices provided that we make the correspondance $\rho \leftrightarrow \omega$ between the density and the vorticity and the correspondance $\Phi \leftrightarrow \psi$ between the potential and the stream function. In that context, the maximization problem (99) associated with the Boltzmann entropy (112) determines the statistical equilibrium state (most probable distribution) of a cloud of point vortices [29, 2].

(ii) *Two-dimensional Euler equation:* The maximization problem (99) determines non-linearly dynamically stable stationary solutions of the 2D Euler-Poisson system

$$(118) \quad \frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = 0,$$

$$(119) \quad \Delta \psi = -\omega,$$

where $\mathbf{u} = -\mathbf{z} \times \nabla \psi$ is the velocity field. The 2D Euler equation describes the evolution of the density distribution of point vortices in the “collisionless” regime (Vlasov regime) before correlations (collisions) have developed [30]. It also describes the inviscid evolution of continuous vorticity flows in two-dimensional hydrodynamics. In that context, $S[\omega]$ is called a H-function. Boltzmann and Tsallis functionals are particular H-functions associated with isothermal and polytropic vortices [3]. Note that the criterion of nonlinear dynamical stability (109) is more refined than the stability criterion (110) which is itself more refined than the Arnol’d theorems [31, 32, 3].

(iii) *Generalized thermodynamics:* The variational principles (99) and (106) determine the generalized thermodynamical stability of complex systems exhibiting anomalous diffusion in physical space. Anomalous diffusion can have various origins: it can be due to the existence of traps, of a lattice preventing particles to reach occupied sites, or any other constraints that are often difficult to formalize. This is what we call “hidden constraints” [6]. These systems can be described by *effective* equations resembling generalized Fokker-Planck equations [3]. The variational principles (99) and (106) determine the linear dynamical stability of stationary solutions of these equations [3].

16. Generalized drift-diffusion equation We shall introduce formally a relaxation equation associated with the maximization problem (106). We write this equation in the form

$$(120) \quad \frac{\partial \rho}{\partial t} = -\nabla \cdot \left[D \rho \nabla \left(\frac{\delta J}{\delta \rho} \right) \right],$$

where $\delta/\delta \rho$ denotes the functional derivative. By construction, this equation conserves mass since the right hand side is the divergence of a current $-\mathbf{J}_\rho$. Using the expression of the free energy (105) with Eqs. (96) and (98), we obtain the generalized drift-diffusion equation

$$(121) \quad \frac{\partial \rho}{\partial t} = \nabla \cdot \left\{ D \left[\rho C''(\rho) \nabla \rho + \beta \rho \nabla \Phi \right] \right\}.$$

Morphologically, this equation can be viewed as a generalized non-local Smoluchowski equation. It describes the dynamics (in physical space) of a system of Langevin particles in interaction governed by a generalized class of stochastic processes [3, 5]:

$$(122) \quad \frac{d\mathbf{r}_i}{dt} = -\xi \nabla_i U(\mathbf{r}_1, \dots, \mathbf{r}_N) + \sqrt{2D\rho_i \left[\frac{C(\rho_i)}{\rho_i} \right]'} \mathbf{R}_i(t),$$

where $\rho_i \equiv \rho(\mathbf{r}_i, t)$ and $\mathbf{R}_i(t)$ is a white noise satisfying $\langle \mathbf{R}_i(t) \rangle = \mathbf{0}$ and $\langle R_{a,i}(t) R_{b,j}(t') \rangle = \delta_{ij} \delta_{ab} \delta(t - t')$, where $a, b = 1, \dots, D$ refer to the coordinates of space and $i, j = 1, \dots, N$ to the particles. The particles interact via the potential $U(\mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{i < j} u(\mathbf{r}_i - \mathbf{r}_j)$ and $C(\rho)$ is an arbitrary convex function. When $C(\rho) = \rho \ln \rho$, which is related to the

Boltzmann entropy (112), Eq. (122) reduces to the usual Langevin equations

$$(123) \quad \frac{d\mathbf{r}_i}{dt} = -\xi \nabla_i U(\mathbf{r}_1, \dots, \mathbf{r}_N) + \sqrt{2D} \mathbf{R}_i(t).$$

This stochastic process describes a system of Brownian particles in interaction. The term in front of $\mathbf{R}_i(t)$ in Eq. (122) can be interpreted as a multiplicative noise since it depends on the position \mathbf{r} . Note that it depends on \mathbf{r} through the density $\rho(\mathbf{r}, t)$. Therefore, there is a back-reaction from the macrodynamics. In that context, the generalized Smoluchowski equation (121) can be obtained from the N -body Fokker-Planck equation by using a Kramers-Moyal expansion and a meanfield approximation [5, 4]. The first term in Eq. (121) is a generalized diffusion (depending on the density) and the second term is a drift. The drift coefficient $\xi = D\beta$ satisfies a generalized Einstein relation. Note that D can depend on \mathbf{r} and t without altering the general properties of the equation. We can use this indetermination to write the generalized drift-diffusion equation in the alternative form

$$(124) \quad \frac{\partial \rho}{\partial t} = \nabla \cdot \left\{ D' \left[\nabla \rho + \frac{\beta}{C''(\rho)} \nabla \Phi \right] \right\},$$

which will have the same general properties as Eq. (121). This equation involves an ordinary diffusion and a nonlinear drift. Equation (124) can be deduced from Eq. (121) by the substitution $D' = D\rho C''(\rho)$. One of these two forms will be preferred depending on the situation contemplated.

The generalized drift-diffusion equation (121) can also be obtained from a variational principle by maximizing the rate of free energy production \dot{J} at fixed mass and temperature [3]. Therefore, Eq. (121) satisfies a canonical H-theorem $\dot{J} \geq 0$. An explicit calculation yields

$$(125) \quad \dot{J} = \int \frac{D}{\rho} \left[\rho C''(\rho) \nabla \rho + \beta \rho \nabla \Phi \right]^2 d^D \mathbf{r} \geq 0.$$

If $\rho \geq 0$, this inequality is true provided that $D > 0$. If ρ can take both signs (which depends on the initial conditions), it is more convenient to consider Eq. (124). In that case, the inequality $\dot{J} \geq 0$ is true provided that $D' > 0$. This shows that J is the Lyapunov functional associated with the generalized drift-diffusion equation (121).

Now, at equilibrium $\dot{J} = 0$, so that according to Eq. (125),

$$(126) \quad \nabla C'(\rho) + \beta \nabla \Phi = \mathbf{0}.$$

Integrating, we get

$$(127) \quad C'(\rho) = -\beta \Phi - \alpha,$$

and we recover Eq. (108). Therefore, a stationary solution of Eq. (121) extremizes the free energy J at fixed mass and temperature. In addition, it is shown in [3] that only *maxima* of J at fixed M and T are linearly stable with respect to the generalized Fokker-Planck equation (121).

Explicating the relation between the potential and the density, the generalized drift-diffusion equation (124) can be written

$$(128) \quad \frac{\partial \rho}{\partial t} = \nabla \cdot \left\{ D \left[\nabla \rho + \frac{\beta}{C''(\rho)} \nabla \int u(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}', t) d^D \mathbf{r}' \right] \right\},$$

with the free energy

$$(129) \quad J = - \int C(\rho) d^D \mathbf{r} - \frac{1}{2} \beta \int \rho \Phi d^D \mathbf{r}.$$

This equation generalizes many drift-diffusion equations introduced in the literature (we refer to [33] for a connexion with mathematical works and for a detailed list of references). It would be of interest to investigate its properties by remaining as general as possible, i.e. without specifying the function $C(\rho)$ and the binary potential of interaction $u(\mathbf{r} - \mathbf{r}')$ explicitly. For example, which conditions must satisfy $C(\rho)$ and $u(\mathbf{r} - \mathbf{r}')$ to generate blow-up solutions? Can we regroup the functionals $S[\rho]$ in “classes of equivalence” with the heuristic idea that functionals of the same “class” will yield “similar” results? Which functionals $S[\rho]$ lead to confined solutions with a compact support, such as polytropic distributions associated with the Tsallis functional [13]? These are interesting mathematical problems which could be tackled in relation with Eq. (128).

17. Relation to Cahn-Hilliard equations. If we now consider the case of short-range interactions, it is possible to expand the potential

$$(130) \quad \Phi(\mathbf{r}, t) = \int u(\mathbf{r}') \rho(\mathbf{r} + \mathbf{r}') d^D \mathbf{r}'$$

in Taylor series for $\mathbf{r}' \rightarrow \mathbf{0}$. Introducing the notations

$$(131) \quad a = \int u(|\mathbf{x}|) d^D \mathbf{x} \quad \text{and} \quad b = \frac{1}{D} \int u(|\mathbf{x}|) x^2 d^D \mathbf{x},$$

we obtain to second order

$$(132) \quad \Phi(\mathbf{r}, t) = a\rho(\mathbf{r}, t) + \frac{b}{2} \Delta \rho(\mathbf{r}, t).$$

In that limit, the free energy takes the form

$$(133) \quad J[\rho] = \frac{1}{2} \beta b \int \left\{ \frac{(\nabla \rho)^2}{2} + V(\rho) \right\} d^D \mathbf{r},$$

where we have set $V(\rho) = -(2/b\beta)C(\rho) - (a/b)\rho^2$. This is the usual expression of the Landau free energy. In general, βb is negative so that we have to minimize the functional integral. For systems with short-range interactions, the conservative equation (128) becomes

$$(134) \quad \frac{\partial \rho}{\partial t} = \nabla \cdot \left\{ \frac{b\xi}{2} \rho \nabla (\Delta \rho - V'(\rho)) \right\}.$$

This is the Cahn-Hilliard equation which has been extensively studied in the theory of phase ordering kinetics. Its stationary solutions describe “domain walls”. We can view therefore Eq. (128) as a generalization of the Cahn-Hilliard equation to the case of systems with long-range interactions. Therefore, its general study is of great mathematical and physical interest.

18. Physical applications The generalized Fokker-Planck equations discussed previously and in [3, 6] can have various physical applications (not necessarily related to thermodynamics) that we briefly mention.

(i) *Numerical algorithms*: The relaxation equations presented in Secs. 7, 8 and 16, and in Refs. [3, 6] can be used as numerical algorithms to solve the variational problems (6), (13), (34), (99) and (106). This is of great practical interest because it is in general difficult to solve the integrodifferential equations (11), (39) and (104) directly and be sure that the solution is a maximum of S or J . These relaxation equations can thus be used as numerical algorithms to construct thermodynamically stable equilibrium states of Hamiltonian systems with long-range interactions (see Sec. 6 (i) and 15 (i)) as well as nonlinearly dynamically stable stationary solutions of the Vlasov, Euler-Jeans and 2D Euler equations (see Secs. 6 (ii), (iii) and 15 (ii)).

(ii) *Violent relaxation*: The relaxation equations introduced in [34, 23, 3, 6] provide a small-scale parameterization of the Vlasov-Poisson and 2D Euler-Poisson systems. They describe the convergence of the flow, on a coarse-grained scale, towards a metaequilibrium state corresponding to a galaxy in astrophysics or a large-scale vortex (e.g., Jupiter's great red spot) in 2D hydrodynamics. The theory of violent relaxation is discussed in, e.g., [22, 23, 35, 2].

(iii) *Statistical mechanics*: In the isothermal case, the nonlocal Kramers and Smoluchowski equations describe the evolution of the distribution function and density of a gas of Brownian particles in interaction [20, 4]. They can be seen, therefore, as the canonical counterpart of the nonlocal Vlasov, Boltzmann and Landau equations describing the evolution of the distribution function of a Hamiltonian system of particles in interaction, for which a microcanonical description holds [5]. Since statistical ensembles are not equivalent for systems with long-range interactions, it is of conceptual interest to compare the microcanonical and canonical descriptions to see the analogies and the differences [19].

(iii) *Biological colonies*: Non-local drift-diffusion equations also occur in biology, in connexion with the chemotactic aggregation of bacterial populations. A model of chemotactic aggregation has been proposed by Keller & Segel [36]. In some approximation [37], their equations reduce to the Smoluchowski-Poisson system as for self-gravitating Brownian particles [19, 20]. This analogy is developed in [4]. In a more general context, the diffusion coefficient or the chemotactic drift can depend on the density. This can take into account anomalous diffusion or finite size effects preventing unphysical blow-up of the bacterial concentration. A simple regularization of the Smoluchowski-Poisson system is provided by the equation

$$(135) \quad \frac{\partial \rho}{\partial t} = \nabla \cdot \left\{ D \left[\nabla \rho + \beta \rho (1 - \rho/\sigma) \nabla \Phi \right] \right\},$$

which is associated with the Fermi-Dirac entropy (114) in physical space. This equation respects the constraint $\rho \leq \sigma$ at any time and therefore prevents blow-up. By rescaling the diffusion coefficient $D(\mathbf{r}, t)$ appropriately, this equation can be put in the form (72) with an effective pressure $p = T \ln(1 - \rho/\sigma)$. A more general model of chemotaxis is provided by the drift-diffusion equation (128) which can take into account anomalous diffusion and finite size effects of various forms. As we have seen, this equation is associated with an effective thermodynamical formalism. An even more general model is represented by the

non-Markovian equation

$$(136) \quad \frac{\partial \rho}{\partial t} = \nabla \cdot \left\{ D \left[\nabla \rho + \frac{\beta}{C''(\rho)} \nabla \int_0^t \int_0^D u(\mathbf{r} - \mathbf{r}', t - t') \rho(\mathbf{r}', t') d^D \mathbf{r}' dt' \right] \right\}.$$

This equation can take into account delay effects that are relevant in the problem of chemotaxis (indeed, the equation satisfied by Φ is of the form $\epsilon \partial_t \Phi = \Delta \Phi - \lambda \rho - \mu \Phi$ [36]). We note that Eq. (136) does not admit a Lyapunov functional (or a free energy) so that the effective thermodynamical formalism developed previously breaks up. It would be of interest to see how these ideas can be generalized to that context.

(iii) *Generalized thermodynamics*: The relaxation equations presented in Secs. 7, 8 and 16, and in Refs. [3, 6] can be considered as *effective* kinetic equations which may be useful to model complex systems. As indicated previously, they can arise when the system is subject to microscopic (hidden) constraints that change the form of the transition probabilities. The consequence is that the accessible microstates are not equiprobable, resulting in new forms of entropic functionals. An explicit and fundamental example is played by the Pauli exclusion principle in quantum mechanics which leads to the Fermi-Dirac entropy instead of the Boltzmann entropy. We have also exhibited a generalized stochastic process (64)-(122) which leads to generalized Fokker-Planck equations of various forms. These stochastic equations form just a particular example of processes generating anomalous diffusion and complex phase space structure. More general and more realistic microscopic processes could also be considered and studied. However, since the generalized kinetic equations (63), (72) and (121) can be obtained from arguments of a very wide scope, such as the Maximum Entropy (Free Energy) Production (Dissipation) Principle [3] for example, we believe that they have a relatively fundamental and universal character and that they will be obtained from a large class of microscopic processes.

19. Conclusion In this paper, we have developed a generalized thermodynamical formalism and corresponding kinetic theories. We believe that generalized thermodynamics is relevant for the physics of complex systems when “hidden constraints” are in action [6]. These microscopic constraints imply that the microstates with given mass and energy are *not* equiprobable, contrary to what is usually *postulated* in thermodynamics. We can either use the Boltzmann entropy and try to take into account these additional microscopic constraints, or consider only the usual constraints (mass and energy) and change the form of entropy [6]. This introduces some indetermination that is encapsulated in the q parameter of Tsallis or more generally in a function $C(f)$. We emphasize that this indetermination is *intrinsic* to the problem and not a flaw of our approach. It occurs because we do not have a complete knowledge of the system’s dynamics. However, we have suggested [3] that generalized entropies can be regrouped in “classes of equivalence” and that, for a given system, a class is more appropriate than another. This extends the attempt of Tsallis and co-workers to try to find the correct value of q corresponding to a given (complex) system. We argue that, more than a scalar q , a full function $C(f)$ must be considered in general.

On the other hand, the maximization problems $\{\text{Max } S \mid E, M \text{ fixed}\}$ and $\{\text{Min } F =$

$E - TS \mid M \text{ fixed}$ can arise in different situations that are not necessarily related to thermodynamics. For example, the maximization of a H function at fixed mass and energy determines nonlinearly dynamically stable stationary solutions of the Vlasov-Poisson system in astrophysics. In that case, this maximization problem is related to dynamics, not thermodynamics. However, in order to investigate this dynamical problem, it can be useful to develop a *thermodynamical analogy* and use a similar vocabulary. We have given other examples (in fluid mechanics, biology,...) where this thermodynamical analogy could be developed.

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